Contents

CHAPTER 1

What are ill-posed problems?

1. Introduction

Definition 1. *A problem of solving an operator equation*

$$Au = f \tag{1.1}$$

where $A : X \longrightarrow Y$ is an operator from a Banach space X into a Banach space Y is called well-posed in the sense of J.Hadamard (1902) iff it satisfies the following conditions.

(1) *(1.1) is solvable for any $f \in Y$. (Existence of the solution, i.e., A is surjective.)*
(2) *The solution to (1.1) is unique. (A is injective.)*
(3) *(1.1) is stable with respect to small perturbations of f. (Continuous dependence of the solution i.e., A^{-1} is continuous.)*

If any of these conditions fails to hold, then problem (1.1) is called ill-posed.

If the solution does not depend continuously on the data, then small errors, whether round off errors, or measurement errors, or perturbations caused by noise, can create large deviations in the solutions. Therefore the numerical treatment of ill-posed problems is a challenge. We shall briefly discuss below some of the concepts and auxiliary results used in this report.

Henceforth $D(A)$, $R(A)$, and $N(A) := \{u : Au = 0\}$ denote the domain, range and null-space of A respectively. Let $A^* : Y^* \longrightarrow X^*$ be the adjoint operator. For simplicity we assume below that $X = Y = H$, where H is a Hilbert space. If A is *self-adjoint* then $A = A^*$. If A is injective, then $N(A) = \{0\}$, A^{-1} is well-defined on $R(A)$ and $u = A^{-1}f$ is the unique solution to equation (1.1) for $f \in R(A)$. Equation (1.1) is called *normally solvable* iff $R(A) = \overline{R(A)}$ i.e., iff $f \perp N(A^*) = \overline{R(A)}^{\perp}$. The overbar denotes closure. If $N(A) \neq \{0\}$ define the *normal solution (pseudo-solution)* u_0 to equation (1.1) as the solution orthogonal to $N(A)$. Then the normal solution is unique and has the property that its norm is minimal: $min\|u\| = \|u_0\|$, where the minimum is taken over the set of all solutions to equation (1.1). The normal solution to the equation $Au = f$ can be defined as the least squares solution: $\|Au - f\| = min., u \perp N(A)$. This solution exists, is unique and depends continuously on f, if H is finite-dimensional. The normal solution is also called *minimal-norm solution.*

A is called *closed* if $\{u_n \longrightarrow u, Au_n \longrightarrow f\}$ implies $\{u \in D(A) \quad and \quad Au = f\}$. By Banach theorem, if A is a linear closed operator defined on all of X then A is bounded. A is called *compact* if it maps bounded sets into pre-compact sets. The set $\{f\}$ is *bounded* means there exists $\rho > 0$ such that $\|f\| \leq \rho$ and a set is *pre-compact* if any subsequence from the set contains a convergent subsequence. In a finite-dimensional Banach space a set is pre-compact iff it is bounded. If A is an injective linear compact operator on an infinite dimensional space then A^{-1} is unbounded.

Singular Value Decomposition: Suppose A is a compact linear operator on H, then A^* is compact and $|A| := [A^*A]^{1/2}$ is self-adjoint, compact and non-negative definite, $|A|\phi_j = \lambda_j \phi_j$, $\lambda_1 \geq \lambda_2 \geq ... \geq 0 \to 0$ are the eigenvalues of $|A|$ with *s-values of* A: $s_j = s_j(A) := \lambda_j(|A|)$ and ϕ_j are the normalized eigenvectors of $|A|$. The faster the s-values go to zero the more ill-posed problem (1.1) is. Any bounded linear operator A admits the polar representation $A := U|A|$, where U is an isometry from $R(A^*)$ onto $R(A)$, $\|Uf\| = \|f\|$, $\|U\| = 1$. One has $|A| = \sum_{j=1}^{\infty} s_j(.,\phi_j)\phi_j$; then the *SVD of* A is $A = U|A| = \sum_{j=1}^{\infty} s_j(.,\phi_j)\psi_j$, where $\psi_j := U\phi_j$.

A closed set K of X is called a *compactum* if any infinite sequence of its elements contains a convergent subsequence. A sequence u_n in U *converges weakly* to u in U iff $\lim_{n\to\infty}(u_n, \phi) = (u, \phi)$ for all $\phi \in H$. We denote the weak convergence by $u_n \rightharpoonup u$. If $u_n \rightharpoonup u$ then $\|u\| \leq \liminf_{n\to\infty} \|u_n\|$. If $u_n \rightharpoonup u$ and A is a bounded linear operator, then $Au_n \rightharpoonup Au$. A bounded set in a Hilbert space contains a weakly convergent subsequence. A functional $F : U \to \mathbb{R}$ is called *convex* if the domain of F is a linear set and for all $u, v \in D(F)$, $F(\lambda u + (1-\lambda)v) \leq \lambda F(u) + (1-\lambda)F(v)$, $0 \leq \lambda \leq 1$.

A functional $F(u)$ is called *weakly lower semicontinuous* from below in a Hilbert space if $u_n \rightharpoonup u$ implies $F(u) \leq \liminf_{n\to\infty} F(u_n)$. A functional $F : U \to \mathbb{R}$ is *strictly convex* if $F(\frac{u_1+u_2}{2}) < \frac{F(u_1)+F(u_2)}{2}$ for all $u_1, u_2 \in D(F)$, provided $u_1 \neq \lambda u_2$, $\lambda =$constant.

Let $F : X \to Y$ be a functional. Suppose $F'(u)\eta = \lim_{\epsilon\to+0} \frac{F(u+\epsilon\eta)-F(u)}{\epsilon}$, exists for all η, and $F'(u)$ is a linear bounded operator in H. Then $F'(u)$ is called *Gateaux* derivative. It is called *Fréchet* derivative if the limit is attained uniformly with respect to η running through the unit sphere.

Spectral Theory: Let A be a self-adjoint operator in a Hilbert space H. To A there corresponds a family E_λ of ortho-projection operators such that $\phi(A) := \int_{-\infty}^{\infty} \phi(\lambda) dE_\lambda$; $\phi(A)f := \int_{-\infty}^{\infty} \phi(\lambda) dE_\lambda f$; $D(\phi(A)) = \{f : \|\phi(A)f\|^2 = \int_{-\infty}^{\infty} |\phi(\lambda)|^2 (dE_\lambda f, f) < \infty\}$; $\|\phi(A)\| = \sup_\lambda |\phi(\lambda)|$.

In particular $A = \int_{-\infty}^{\infty} \lambda dE_\lambda$ and $D(A) = \{f : \|Af\|^2 = \int_{-\infty}^{\infty} |\lambda|^2 (dE_\lambda f, f) < \infty\}$. E_λ is called the *resolution of the identity* corresponding to the self-adjoint operator A. λ is taken over the spectrum of A. We have $\rho(\lambda) = (E_\lambda f, f)$; $(dE_\lambda f, f) = d(E_\lambda f, f) = d\rho(\lambda)$; $I = \int_{-\infty}^{\infty} dE_\lambda$; $f = \int_{-\infty}^{\infty} dE_\lambda f$; $\|f\|^2 = \int_0^{\|A\|} d(E_\lambda f, f)$; $E_{+\infty} = I$; $E_{-\infty} = 0$; $E_{\lambda-0} = E_\lambda$.

Let the operator equation $Au = f$ be solvable (possibly non-uniquely). Let y be its minimal-norm solution, $y \perp N(A)$. Let $B = A^*A \geq 0$ and $q := A^*f$. Then $Bu = q$. Also,

$(B+\alpha)^{-1} := (B+\alpha I)^{-1}$ is a positive definite operator and is given by

$(B+\alpha)^{-1} = \int_0^{\infty} \frac{dE_\lambda}{\lambda+\alpha}$.

$\|(B+\alpha)^{-1}\| = \sup_{0\leq\lambda\leq\|B\|} |\frac{1}{\lambda+\alpha}| \leq \frac{1}{\alpha}, \alpha > 0$.

$\|[(A+\alpha)^{-1}A - I]f\|^2 = \alpha^2 \|(A+\alpha)^{-1}f\|^2 = \alpha^2 \int_0^{\|A\|} \frac{d(E_\lambda f, f)}{(\lambda+\alpha)^2} \leq \int_0^{\|A\|} d(E_\lambda f, f) = \|f\|^2 < \infty$.

If $\alpha \to 0$, the integrand $\frac{\alpha^2}{(\lambda+\alpha)^2}$ tends to zero. So, by the Lebesgue dominated convergence theorem $\|[(A+\alpha)^{-1}A - I]f\|^2 \to 0$ as $\delta \to 0$, provided that $\int_0^{0^+} d(E_\lambda f, f) = 0$, i.e., $f \perp N(A)$.

LEMMA 1. *If the equation $Au = f$ is solvable then it is equivalent to the equation $Bu = q$.*

Proof: $(\Rightarrow)Au = f$, so $A^*Au = A^*f$.

$(\Leftarrow)A^*Au = A^*Ay$, so $A^*A(u-y) = 0$, hence $A(u-y) = 0$ and hence $Au = Ay = f$. Thus we have proved the lemma. □

The mapping $A^+ : f \longrightarrow u_0$ is called the *pseudo-inverse* of A. A^+ is a bounded linear operator iff it is normally solvable and $R(A)$ is closed. So equation (1.1) is ill-posed iff A^+ is unbounded. One can find the details of this in [**5**].

An operator $\mathbf{\Phi(t,u)}$ is *locally Lipschitz* with respect to $u \in H$ in the sense $sup\|\mathbf{\Phi(t,u)} - \mathbf{\Phi(t,v)}\| \leq c\|u-v\|, c = c(R, u_0, T) > 0$ where the supremum is taken for all $u, v \in B(u_0, R)$, and $t \in [0, T]$.

2. Examples of ill-posed problems

Example 1. Stable numerical differentiation of noisy data

The problem of numerical differentiation is ill-posed in the sense that small perturbations of the function to be differentiated may lead to large errors in its derivative. Let $f \in C^1[0,1]$, with noisy data $\{\delta, f_\delta\}$, where $\delta > 0$ is the noise level, that is we have the estimate

$$\|f_\delta - f\| \le \delta. \tag{2.1}$$

The problem is to estimate stably the derivative f', i.e., to find such an operation R_δ such that the error estimate
$\|R_\delta f_\delta - f'\| \le \eta(\delta) \longrightarrow 0$ as $\delta \longrightarrow 0$.
This problem is equivalent to stably solving the equation

$$Au := \int_0^x u(t)dt = f(x), \quad A : H := L^2[0,1] \longrightarrow L^2[0,1]; \quad f(0) = 0, \tag{2.2}$$

if noisy data f_δ are given in place of f. In this case, finding $f' = A^{-1}f$, given the data f_δ is an ill-posed problem, since equation (2.2) may have no solution in $L^2[0,1]$ if $f_\delta \in L^2[0,1]$ is arbitrary, subject to only the restriction $\|f_\delta - f\| \le \delta$, and if $f_\delta \in C^1[0,1]$ then f'_δ may differ from f' as much as one wishes however small δ is. Also, if A is a linear compact operator then A^{-1}, if it exists is unbounded and hence equation (2.2) is ill-posed. The problem is: given $\{\delta, A, f_\delta\}$, find a *stable approximation* u_δ to the solution $u(x) = f'(x)$ of the equation (2.2) in the sense the error estimate

$$\|u_\delta - u\| \le \eta(\delta) \longrightarrow 0 \quad as \quad \delta \longrightarrow 0. \tag{2.3}$$

For this we try to construct an operator $R_\alpha : H \longrightarrow H$ such that

$$u_\delta := R_{\alpha(\delta)} f_\delta \tag{2.4}$$

satisfies the error estimate (2.3). R_α depends on a parameter α and is called a *regularizer* if R_α is applicable to any $f_\delta \in Y$ and if there is a choice of the regularizing parameter $\alpha \equiv \alpha(\delta) \to 0$ as $\delta \to 0$ such that
$R_\delta f_\delta := R_{\alpha(\delta)} f_\delta \longrightarrow u \quad as \quad \delta \longrightarrow 0.$

Example 2. The Cauchy problem for the Laplace equation

We consider the classical problem posed by J.Hadamard. It is required to find the solution $u(x,y)$ of the Laplace equation

$$u_{xx} + u_{yy} = 0 \tag{2.5}$$

in the domain $\Omega = \{(x,y) \in R^2 : y > 0\}$ satisfying the boundary conditions

$$u(x,0) = 0, \quad u_y(x,0) = \phi(x) = A_n \sin nx; \quad A_n \longrightarrow 0 \quad as \quad n \longrightarrow \infty. \tag{2.6}$$

The *Cauchy problem* consists of finding a solution of the equation (2.5) satisfying the conditions (2.6). The data differ from zero as little in the sup-norm as can be wished, if n is sufficiently large. Its solution is given by

$$u(x,y) = \frac{A_n}{n} \sin(nx) \sinh(ny), \tag{2.7}$$

which, if $A_n = 1/n$, is very large for any value of $y > 0$, because $\sinh(ny) = 0(e^{ny})$. As $n \longrightarrow \infty$, the Cauchy data tend to zero in $C^1(\mathbb{R})$, and $u \equiv 0$ is a solution to equation (2.5) with $u = u_y = 0 \quad at \quad y = 0$. Thus, even though the solution to the

Cauchy problem (2.5)-(2.6) is unique, continuous dependence of the solution on the data in the sup-norm does not hold. This shows that the Cauchy problem for the Laplace equation (2.5) is an ill-posed problem.

Example 3. Fredholm integral equations of the first kind

Consider the problem of finding the solution to the integral equation

$$Au(x) = \int_0^1 K(x,y)u(y)dy = f(x), \quad 0 \leq x \leq 1 \tag{2.8}$$

where the operator $A : H := L^2(0,1) \longrightarrow L^2(0,1)$ is compact and > 0 almost everywhere, with kernel $K(x,y)$ satisfying the condition:

$$\int_0^1 \int_0^1 |K(x,y)|^2 dxdy < \infty. \tag{2.9}$$

Then $A : H \longrightarrow H$ is compact. A compact operator in an infinite-dimensional space cannot have a bounded inverse. That is the problem (2.8) is ill-posed.

3. Regularizing family

Consider the operator equation given by (1.1) with the following assumptions:

(1) A is not continuously invertible.
(2) For exact values of f and A, there exists a solution u of equation (1.1).
(3) A is known exactly, and instead of f, we are given its approximation $f_\delta \in Y$ such that the estimate (2.1) is satisfied in Y.

where $\delta > 0$ is a numeric parameter characterizing the errors of input data f_δ.

We need a numerical algorithm for solving the operator equation satisfying the condition that the smaller the value of δ , the closer the approximation to u is obtained, i.e., the error estimate (2.3) is satisfied. The *Regularizing Algorithm (RA)* is the operator $R_\alpha : Y \longrightarrow X$ which, for a suitable choice of $\alpha \equiv \alpha(\delta)$, puts into correspondence to any pair $\{\delta, f_\delta\}$, the element $u_\delta \in X$ such that the error estimate (2.3) is satisfied where $u_\delta := R_{\alpha(\delta)} f_\delta$. For a given set of data, u_δ is the approximate solution of the problem. Based on the existence and construction of RA, all ill-posed problems may be classified into regularizable (i.e., the ones for which a RA exists) and non-regularizable, and *solving an ill-posed problem means constructing RA for such a problem.*

Let $Au = f$, where $A : X \longrightarrow Y$ is a linear injective operator, $R(A) \neq \overline{R(A)}$, $f \in R(A)$ is not known and the data are the elements f_δ such that the estimate (2.1) is satisfied. The objective is to find stable approximation u_δ to the solution u such that the error estimate (2.3) is satisfied. Such a sequence u_δ is called a *stable solution* to the equation (1.1) with the perturbed (or noisy) data.

Let the operator equation $Au = f$, be given, and f_δ, $\|f_\delta - f\| \leq \delta$, the noisy data be given in place of f. Let A be injective. Then R_α is called a *regularizer* of the operator equation if $D(R_\alpha) = H$ and there exists $\alpha(\delta) \to 0, \quad \delta \to 0$ such that $\|R_{\alpha(\delta)} f_\delta - u\| \to 0$ as $\delta \to 0$ for all $u \in H$.

Lemma 2. *If there exists an operator R_α, $D(R_\alpha) = H$, $\|R_\alpha\| \leq a(\alpha)$, such that $\|R_\alpha Au - u\| := \eta(\alpha) \to 0$ as $\alpha \to 0$, and the function $g(\alpha) := -\frac{\eta'(\alpha)}{a'(\alpha)}$ is monotone for $\alpha \in (0, \alpha_0)$, $\alpha_0 > 0$ a small number with $g(+0) = 0$, then there exists an $\alpha \equiv \alpha(\delta) \to 0$ as $\delta \to 0$ such that $\|R_{\alpha(\delta)} f_\delta - u\| \to 0$ as $\delta \to 0$ for all $u \in H$.*

Proof: Consider, $\|R_\alpha f_\delta - u\| \leq \|R_\alpha(f_\delta - f)\| + \|R_\alpha f - u\| \leq \delta a(\alpha) + \eta(\alpha)$ and $\alpha(\delta)$ can be chosen suitably. One can choose $\alpha(\delta)$ so that the minimization problem

$\delta a(\alpha) + \eta(\alpha) = min. \longrightarrow 0$ as $\alpha \longrightarrow 0$. (*)

Equation, $\delta a'(\alpha) + \eta'(\alpha) = 0$. (**) is a necessary condition for min. in (*). Since problem (1.1) is ill-posed, one has $a(\alpha) \to \infty$ as $\alpha \to 0$. The function $a(\alpha)$ can be assumed monotone decreasing and $\eta(\alpha)$ can be assumed monotone increasing on $(0, \alpha_0)$, $\eta(0) = 0$. We assume in lemma, $\eta(\alpha) \to 0$, as $\alpha \to 0$ $[\eta(\alpha) > 0$, $,a'(\alpha) < 0, \eta'(\alpha) > 0, \alpha > 0]$. Since $g(\alpha) := -\frac{\eta'(\alpha)}{a'(\alpha)}$ is a monotone function for $\alpha \in (0, \alpha_0)$, equation (*) has a unique solution $\alpha \equiv \alpha(\delta)$ for any sufficiently small $\delta > 0$.

More precisely for any fixed f_δ, an operator $R_\delta : H \longrightarrow H$ such that $R_\delta := R_{\alpha(\delta),\delta}$, which depends on δ is a regularizer for equation (1.1), if for some choice of $\alpha, \quad \alpha \equiv \alpha(\delta)$, one has,

$$\|R_\delta f_\delta - A^{-1} f\| \longrightarrow 0 \quad as \quad \delta \longrightarrow 0, \quad for \quad all \quad f \in R(A). \tag{3.1}$$

so that u is stably approximated. $\square$

If such a family R_δ is known then the function

$$u_\delta := R_\delta f_\delta, \tag{3.2}$$

satisfies the error estimate (2.3) in view of (3.1), i.e., formula (3.2) gives a stable approximation to the solution u of equation (1.1). The scalar parameter α is called the *regularization parameter*.

A construction of a family $R_{\alpha,\delta}$ of operators such that there exists a unique $\alpha(\delta)$ satisfying (3.1) is always used for solving an ill-posed problem (1.1). The operator A^{-1} is said to be *regularizable*, if there exists a regularizer R_δ which approximates A^{-1} in the sense of (3.1) using the noisy data $\{\delta, f_\delta\}$. In the case of well-posed problems, A^{-1} is always regularizable: one may take $R_\delta = A^{-1}$ for all δ. This can happen only for well-posed problem. If the problem is ill-posed then there does not exist a regularizer independent of the noise δ.

Example: In the stable numerical differentiation example, we shall take

$$Au(x) = \int_0^x u(s)ds = f(x), \quad 0 \le x \le 1. \tag{3.3}$$

Suppose $f_\delta(x)$ is given in place of f:

$$\|f - f_\delta\|_{L^\infty(0,1)} \le \delta. \tag{3.4}$$

Following A.G.Ramm [**17**], we choose a regularizer R_δ of the form:

$$u_\delta = R_\delta f_\delta := \frac{f_\delta(x + h(\delta)) - f_\delta(x - h(\delta))}{2h}, \quad h(\delta) = \sqrt{\frac{2\delta}{M}} > 0. \tag{3.5}$$

We note that A.G.Ramm [**3**] has given a new notion of regularizer. According to [**3**] a family of operators $R(\delta)$ is a *regularizer* if
$sup\|R(\delta)f_\delta - v\| \le \eta(\delta) \longrightarrow 0 \quad as \quad \delta \longrightarrow 0$,
where the supremum is taken over all $v \in S_\delta = \{v : \|Av - f_\delta\| \le \delta, v \in K\}$, and K is a compactum in X to which the solution u belongs. The difference between Ramm's definition and the original definition is that in the original definition u is fixed, one does not know the solution $u \in K$, the only information available is a family f_δ and some apriori information about the solution u, while in the new definition v is any element of S_δ and the supremum, over all such v, of the norm above must go to zero as δ goes to zero. This definition is more natural in the sense that not only the solution u to (1.1) satisfies the estimate $\|Au - f_\delta\| \le \delta$, but many $v \in K$ satisfy such an inequality $\|Av - f_\delta\| \le \delta, v \in K$, and the data f_δ may correspond, to any $v \in S_\delta$, and not only to the solution of problem (1.1).

CHAPTER 2

Review of the methods for solving ill-posed problems

In this chapter, we shall discuss four different methods, (variational regularization method, quasi-solutions method, iterative regularization method and dynamical systems method) for constructing regularizing families for ill-posed problems (1.1) with bounded operators. See also A.G.Ramm[**1**] for ill-posed problems with unbounded operators.

1. Variational regularization method

This method consists of solving a variational problem, which was proposed by D.Phillips (1962) and studied by A.N.Tikhonov (1963) et al by constructing regularizers for solving ill-posed problems.

Consider equation (1.1), which has to be solved, where $A : X \longrightarrow Y$ is assumed to be a bounded, linear, injective operator, with A^{-1} not continuous, $f \in R(A)$ is not known, and the data are the elements $\{\delta, A, f_\delta\}$, where the noise level $\delta > 0$ is given, estimate (2.1) holds, and the noisy data f_δ is the $\delta-$approximation of f. The problem is: given $\{\delta, A, f_\delta\}$, find the stable solution u_δ such that the error estimate (2.3) holds.

Let equation (1.1) have a minimal-norm solution y. Variational regularization method consists of solving the variational problem (minimization problem) and constructing a stable approximation to solution y with minimal-norm such that $y \perp N(A)$. Assume without loss of generality $\|A\| \leq 1$, and then $\|A^*\| \leq 1$. Let $B := A^*A$, then $B \geq 0$ and is a bounded, self-adjoint operator. The equation $Au = f$ is equivalent to the equation $Bu = q$, where $q := A^*f$. Assume that q_δ is given in place of q, $\|q - q_\delta\| \leq \|A^*\|\delta \leq \delta$. Since $N(A) = N(B)$, $y \perp N(B)$ and $\|(B + \alpha)^{-1}\| \leq \frac{1}{\alpha}$.

Consider the problem of finding the minimum of the functional

$$F(u) := \|Au - f_\delta\|^2 + \alpha\|u\|^2 = min. \tag{1.1}$$

where $\alpha > 0$ is the regularization parameter. The functional $F(u)$ is a function of two parameters α and δ. Solutions of variational problem (1.1) are called *minimizers*. First, we shall prove the following two lemmas.

Lemma 3. Existence of minimizers: *For arbitrary $\alpha>0$ and $\delta>0$, there exists a solution $u_{\alpha,\delta}$ to variational problem (1.1), in the sense $F(u_{\alpha,\delta}) \leq F(u)$ for all $u \in X$.*

Lemma 4. Uniqueness of minimizers: *The solution $u_{\alpha,\delta}$ of variational problem (1.1) is unique.*

Proof of Lemma 3: Define
$m := \inf_{u\in H} F(u)$. Note that $m \equiv m(\delta) \geq 0$, and $m \leq F(u), u \in H$.
Let $\{u_n\} \in D(F)$ be a minimizing sequence for the functional F, such that

$$m \leq F(u_n) \leq m + \epsilon_n, \quad \epsilon_n \longrightarrow 0 \quad as \quad n \longrightarrow \infty. \tag{1.2}$$

So, $\alpha\|u_n\|^2 \leq m + \epsilon_n$. Hence $\|u_n\|^2 \leq \frac{m+\epsilon_n}{\alpha}$.

Since a bounded set in H contains a weakly convergent subsequence, there exists a weakly convergent subsequence of $\{u_n\}$, denoted again by $\{u_n\}$, with $u_n \rightharpoonup u$. This implies by continuity of A, $Au_n \rightharpoonup Au$. Thus,

$\|u\| \leq \liminf_{n\to\infty} \|u_n\|$ and $\|Au - f_\delta\|^2 \leq \liminf_{n\to\infty} \|Au_n - f_\delta\|^2$.

So from equation (1.2) we obtain
$m \leq \|Au - f_\delta\|^2 + \alpha\|u\|^2 \leq \liminf_{n\to\infty}(\|Au_n - f_\delta\|^2 + \|u_n\|^2) \leq \liminf_{n\to\infty} F(u_n)$
$\leq \lim_{n\to\infty}(m + \epsilon_n) = m$.

So we have $m \leq F(u) \leq m$. Hence $F(u) = m$. So u is the minimizer of $F(u)$. Thus we have proved the existence of the minimizer for the variational problem (1.1). □

Proof of Lemma 4: Since u is the minimizer of variational problem (1.1), it follows that $F(u) \leq F(u + \epsilon\eta)$, for any $\eta \in H$, and for any $\epsilon \in (0, \epsilon_0)$. So, $\lim_{\epsilon\to 0} \frac{F(u+\epsilon\eta)-F(u)}{\epsilon} \geq 0$.

Assuming that $F'(u)$ exists, this implies that $(F'(u), \eta) \geq 0$ for all $\eta \in H$ and hence $F'(u) = 0$. We shall calculate the derivative of (1.1) with respect to ϵ at $\epsilon = 0$ and get:

$\frac{d}{d\epsilon}F(u + \epsilon\eta)|_{\epsilon=0} = [\frac{d}{d\epsilon}\|Au + A\epsilon\eta - f_\delta\|^2 + \alpha\|u + \epsilon\eta\|^2]|_{\epsilon=0}$.

Since

$[\frac{d}{d\epsilon}\|u + \epsilon\eta\|^2]|_{\epsilon=0} = [\frac{d}{d\epsilon}(u + \epsilon\eta, u + \epsilon\eta)]|_{\epsilon=0} = [(\eta, u + \epsilon\eta), (u + \epsilon\eta, \eta)]|_{\epsilon=0} = (\eta, u) + (u, \eta) = 2Re(u, \eta)$

and

$[\frac{d}{d\epsilon}\|Au + A\epsilon\eta - f_\delta\|^2]|_{\epsilon=0} = [\frac{d}{d\epsilon}(Au + \epsilon A\eta - f_\delta, Au + \epsilon A\eta - f_\delta)]|\epsilon=0 = (A\eta, Au - f_\delta) + (Au - f_\delta, A\eta) = 2Re(A^*Au - A^*f_\delta, \eta)$.

So,
$\frac{d}{d\epsilon}F(u + \epsilon\eta)_{|\epsilon=0} = 2Re(A^*Au - A^*f_\delta + \alpha u, \eta) = 0$
for all $\eta \in H$. Hence we obtain,

$$A^*Au + \alpha u = A^*f_\delta. \tag{1.3}$$

Thus if u is a minimizer of $F(u)$, then equation (1.3) holds. We claim that equation (1.3) has not more than one solution. For this, it is sufficient to prove that $A^*Aw + \alpha w = 0$ implies $w = 0$. Suppose that

$$A^*Aw + \alpha w = 0, \quad \alpha = constant > 0. \tag{1.4}$$

then $0 = ((A^*A + \alpha)w, w)$
$= (A^*Aw, w) + \alpha(w, w) = (Aw, Aw) + \alpha(w, w) = \|Aw\|^2 + \alpha\|w\|^2 \geq \alpha\|w\|^2, \quad \alpha > 0.$

Therefore, $w = 0$. Hence the solution to equation (1.3) is unique, and is given by the formula

$$u_{\alpha,\delta} := (A^*A + \alpha)^{-1}A^*f_\delta. \tag{1.5}$$

for every $\alpha > 0$ and the operator $(A^*A + \alpha)^{-1}$ exists and is bounded by $\|(A^*A + \alpha)^{-1}\| \leq \frac{1}{\alpha}$, because $A^*A \geq 0$. □

We shall now consider the main theorem which gives us a method for constructing a regularizing family for the ill-posed problem (1.1).

THEOREM 1. *Assume that A is a linear bounded, injective operator, equation $Au = f$ is solvable, A^{-1} is not continuous, and let y be the minimal-norm solution: $Ay = f$, $y \perp N(A)$. If f_δ is given in place of f, $\|f - f_\delta\| \leq \delta$. Then for any $\alpha > 0$, minimization problem (1.1) has a unique solution $u_{\alpha,\delta}$ given by the formula (1.5). Moreover, if $\alpha \equiv \alpha(\delta)$ is such that*

$$\alpha(\delta) \longrightarrow 0 \quad and \quad \frac{\delta}{\alpha(\delta)} \longrightarrow 0 \quad as \quad \delta \longrightarrow 0, \tag{1.6}$$

then $u_\delta := u_{\alpha(\delta),\delta} \longrightarrow y \quad as \quad \delta \longrightarrow 0$.

Proof: The proofs of existence and uniqueness of the minimizers were given above. Let us prove the last conclusion of the theorem. Assume that condition (1.6) holds.

Define the regularizer (by means of formula (1.5) so that it satisfies equation (1.1)):

$$R_\alpha f_\delta := R_{\alpha(\delta)} f_\delta := u_{\alpha,\delta} = (B + \alpha)^{-1} A^* f_\delta, \quad \alpha = \alpha(\delta). \tag{1.7}$$

If $f = Ay$, then $R_\alpha Ay = (B + \alpha)^{-1} By \longrightarrow y$ as $\alpha \longrightarrow 0$. Here the assumption $y \perp N(A)$ is used. We claim that

$$\|R_\alpha f_\delta - y\| \longrightarrow 0 \quad as \quad \alpha \longrightarrow 0. \tag{1.8}$$

Since,

$\|R_\alpha f_\delta - y\| \leq \|R_\alpha(f_\delta - f)\| + \|R_\alpha f - y\| \leq \|f_\delta - f\| \|R_\alpha\| + \eta(\alpha) \leq \frac{\delta}{\alpha}\|A^*\| + \eta(\alpha)$ $\leq \frac{\delta}{\alpha} + \eta(\alpha) \to 0$ as $\delta \longrightarrow 0$, because of (1.6). □

2. Discrepancy principle for variational regularization method

Assumptions (A) Let A be a linear, bounded, injective operator. Let equation $Au = f$ be solvable. Let A^{-1} not be continuous and let there exist a minimal-norm solution y such that $Ay = f$, $y \perp N(A)$. Let f_δ be given in place of f, $\|f - f_\delta\| \leq \delta$. Let u_δ be the stable solution of equation (1.1) given by $u_\delta = (B + \alpha)^{-1} A^* f_\delta$.

The discrepancy principle (DP) introduced by Morozov is used as an aposteriori choice of the regularization parameter α and this choice yields convergence of the variational regularization method. Choose the regularization parameter α=$\alpha(\delta)$ as the root of the equation

$\|Au_{\alpha,\delta} - f_\delta\| = C\delta, \quad C = constant > 1. \qquad (*)$

The above equation, is a non-linear equation with respect to α. It defines α as an implicit function of δ. Let us assume that $\|f_\delta\| > C\delta$.

THEOREM 2. *Suppose that the assumptions (A) above holds. Then there exists a unique solution* α= $\alpha(\delta) > 0$ *to equation (*) and* $\lim_{\delta\to 0} \alpha(\delta) = 0$. *Moreover, if* $u_\delta := u_{\alpha(\delta),\delta}$ *is given by formula (1.7), then* $\lim_{\delta\to 0} \|u_\delta - y\| = 0$.

Proof: Denote $Q := AA^*$. Then, $N(Q) = N(A^*)$, and $(B + \alpha)^{-1} A^* = A^*(Q + \alpha)^{-1}$. By variational regularization method, for any $\alpha > 0$, minimization problem (1.1) has a unique solution $u_{\alpha,\delta} = (B + \alpha)^{-1} A^* f_\delta$. Since, $\|f_\delta - f\| \leq \delta$; $f = Ay$, so, $f \perp N(A^*)$. From equation (*),

$C^2\delta^2 = \|Au_{\alpha,\delta} - f_\delta\|^2 = \|[A(B + \alpha)^{-1} A^* - I]f_\delta\|^2 = \|[Q(Q + \alpha)^{-1} - I]f_\delta\|^2$

$= \alpha^2 \|(Q + \alpha)^{-1} f_\delta\|^2 = \alpha^2 \int_0^{\|Q\|} \frac{d(E_\lambda f_\delta, f_\delta)}{(\lambda+\alpha)^2} := I(\alpha, \delta)$

one has,

$\lim_{\alpha\to\infty} I(\alpha, \delta) = \int_0^{\|Q\|} d(E_\lambda f_\delta, f_\delta) = \|f_\delta\|^2 > C^2\delta^2.$

and

$\lim_{\alpha\to+0} I(\alpha, \delta) = \|P_{N(Q)} f_\delta\|^2.$

where $P_{N(Q)}$ is the orthogonal projection onto the null-space of Q,

$P_{(a,b)} f_\delta := \int_a^b dE_\lambda f_\delta$

and

$\|P_{(a,b)} f_\delta\|^2 := \int_a^b d(E_\lambda f_\delta, f_\delta).$

So

$\|P_{N(Q)} f_\delta\|^2 \leq \int_0^\epsilon \frac{\alpha^2 d(E_\lambda f_\delta, f_\delta)}{(\alpha+\lambda)^2} \leq \int_0^\epsilon d(E_\lambda f_\delta, f_\delta) = \|P_{(0,\epsilon)} f_\delta\|^2.$

Since

$P_{N(A^*)} f_\delta = P_{N(A^*)}(f_\delta - f) + P_{N(A^*)} f$

and since $P_{N(A^*)} f = 0$, one has

$\|P_{N(A^*)} f_\delta\|^2 = \|P_{N(A^*)}(f_\delta - f)\|^2 \leq \|f_\delta - f\|^2 \leq \delta^2.$

Thus,

$\lim_{\alpha\to+0} I(\alpha, \delta) = \|P_{N(Q)} f_\delta\|^2 = \|P_{N(A^*)} f_\delta\|^2 \leq \delta^2 < C^2\delta^2.$

Equation (*) is a non-linear equation of the form $C^2\delta^2 = I(\alpha, \delta)$, for a given fixed pair $\{f_\delta, \delta\}$, the function $I(\alpha, \delta)$ satisfies $\lim_{\alpha\to+0} I(\alpha, \delta) < C^2\delta^2$ and $\lim_{\alpha\to\infty} I(\alpha, \delta) > C^2\delta^2$. Hence $I(\alpha, \delta)$ is a monotone increasing function of α on $(0, \infty)$. Hence, equation (*) has a unique solution α=$\alpha(\delta)$. Now let us prove that $\lim_{\delta\to 0} \alpha(\delta) = 0$. Suppose that $\alpha(\delta) \geq \alpha_0 > 0$. So as $\delta \to 0$,

$0 \leftarrow C^2\delta^2 = \int_0^{\|Q\|} \frac{\alpha^2 d(E_\lambda f_\delta, f_\delta)}{(\alpha+\lambda)^2} \geq \int_0^{\|Q\|} \frac{\alpha_0{}^2 d(E_\lambda f_\delta, f_\delta)}{(\alpha_0+\lambda)^2} \geq \frac{\alpha_0{}^2}{(\alpha_0+\|Q\|)^2} \|f_\delta\|^2 > 0.$

This contradicts the assumption that $\alpha(\delta) > 0$. It remains to prove the last conclusion of the theorem. Define the regularizer by the formula (by means of formula (1.7))

$R_\delta f_\delta := u_\delta := u_{\alpha(\delta),\delta}$,
where $\alpha(\delta)$ is given by the discrepancy principle. Let us prove that $\|u_\delta - y\| \to 0$ as $\delta \to 0$. Since u_δ is a minimizer of (1.1), we have
$F(u_\delta) \leq F(u), \quad \|Au_\delta - f_\delta\|^2 + \alpha(\delta)\|u_\delta\|^2 \leq \delta^2 + \alpha(\delta)\|y\|^2 \qquad (**)$
then from equations (*) and (**), we obtain,
$\|u_\delta\|^2 \leq \|y\|^2, \quad \|u_\delta\| \leq \|y\|$.

This implies that there exists v such that $u_\delta \rightharpoonup v \quad as \quad \delta \longrightarrow 0$ and by continuity of A, $Au_\delta \longrightarrow Av$. So from (**), as $\delta \to 0$ and $\alpha \to 0$ it follows that $Au_\delta \longrightarrow f$. So, $Av = f$. Since A is injective, this implies that $v = y$. So, $u_\delta \rightharpoonup y \quad as \quad \delta \longrightarrow 0$. Also, since, $\|u_\delta\| \leq \|y\|$,
$\|y\| \leq \liminf_{\delta\to 0} \|u_\delta\| \leq \limsup_{\delta\to 0} \|u_\delta\| \leq \|y\|$.
Therefore, $\lim_{\delta\to 0} \|u_\delta\|$ exists and $\lim_{\delta\to 0} \|u_\delta\| = \|y\|$. Thus, $\lim_{\delta\to 0} \|u_\delta - y\| = 0$. Hence the theorem is proved. □

Note that: $F(P_{N(A)^\perp} u) \leq F(u)$.

Proof: Let $u = u_0 + u_1$, where $u_0 \in N(A)$ and $u_1 \in N(A)^\perp$.

$Au = Au_0 + Au_1 = Au_1$, since $Au_0 = 0$.

So, $\|Au - f_\delta\|^2 = \|Au_1 - f_\delta\|^2$.
Also, $\alpha\|u\|^2 = \alpha[\|u_0\|^2 + \|u_1\|^2] \geq \alpha\|u_1\|^2$.

This implies that, $F(u_1) \leq F(u)$. So $F(u_1) = F(P_{N(A)^\perp} u) \leq F(u)$. Hence a minimizer of F is necessarily orthogonal to null-space of A.

Remark: A.G.Ramm [**7**] has generalized the discrepancy principle for the cases: (a) when A is not injective, (b) when A is not compact and not injective and (c) when A^{-1} is not continuous. He has also shown that discrepancy principle, in general does not yield convergence which is uniform with respect to the data.

3. The method of quasi-solution

The method of quasi-solution was given by Ivanov (1962). It is similar to the variational regularization method except that there is a restriction on the functional defined.

Consider the operator equation (1.1) which has to be solved, where A is assumed to be a bounded, linear injective operator on Banach spaces X and Y or $R(A)$ is assumed not to be closed, so that the problem is ill-posed. The data are the elements $\{\delta, A, f_\delta\}$, where the noise level $\delta > 0$ is given such that the estimate (2.1) holds, i.e., the noisy data f_δ is the $\delta-$approximation of f. The problem is: given $\{\delta, A, f_\delta\}$, find the stable solution u_δ such that the error estimate (2.3) holds. Let equation (1.1) have a solution $y \in K$, a convex compactum (closed, pre-compact subset) of X. Consider the variational problem:

$$F(u) := \|Au - f_\delta\| \longrightarrow inf., \quad u \in K. \tag{3.1}$$

Definition 2. *A quasi-solution of equation (1.1) on a compactum K is a solution to the minimization problem (3.1).*

Lemma 5. *: Existence of quasi-solution:*
Assume that A is a bounded linear injective operator and that equation (1.1) holds. Assume that equation (1.1) has a solution $y \in K$ a compactum of X. Then the minimization problem (3.1) has a stable solution $u_\delta \in K$ such that $\|u_\delta - y\| \longrightarrow 0 \quad as \quad \delta \longrightarrow 0$.

Proof: Denote

$$m(\delta) := \inf_{u \in K} \|Au - f_\delta\|. \tag{3.2}$$

Since, the infimum $m = m(\delta)$ depends on f_δ and since $y \in K$, we have,
$m(\delta) = \inf_{u\in K} \|Au - f_\delta\| \leq \|Ay - f_\delta\| = \|f - f_\delta\| \leq \delta$.
So $m(\delta) \longrightarrow 0 \quad as \quad \delta \longrightarrow 0$. Let u_n be a minimizing sequence in K:

$$F(u_n) := \|Au_n - f_\delta\| \longrightarrow m(\delta), \quad u_n \in K, \quad n \longrightarrow \infty. \tag{3.3}$$

So we have $\sup_n \|Au_n\| < \infty$. Let us now take $\delta \longrightarrow 0$. Since $u_n \in K$ and K is a compactum, there exists a convergent subsequence in K, which we again denote by u_n, such that $u_n \longrightarrow u_\infty$. Since K is a compactum, it is closed. Therefore the limit $u_\infty \in K$. By continuity of A, this implies that $Au_n \longrightarrow Au_\infty$, and $\lim_{n\to\infty} \|Au_n - f_\delta\| = \|Au_\infty - f_\delta\| = m(\delta)$.

Denote

$$u_\delta := u_\infty \equiv u_\infty(\delta) \in K. \tag{3.4}$$

Thus, u_δ is the solution of the minimization problem (3.1):
$\|Au_\delta - f_\delta\| = m(\delta)$.

It remains to be shown that $u_\delta \in K$, is the quasi-solution of equation (1.1). Now, as $\delta \longrightarrow 0$, there exists a subsequence $u_{\delta_n} \in K$ which is again denoted by u_n, such that $u_n \longrightarrow v \in K$. By continuity of A, this implies that $Au_n \longrightarrow Av$. Therefore, since $m(\delta) \longrightarrow 0 \quad as \quad \delta \longrightarrow 0$,

$$\|Av - f\| \longrightarrow 0, \quad as \quad \delta \longrightarrow 0. \tag{3.5}$$

Since, A is injective, $v = y$. Thus,

$$\lim_{\delta\to 0} \|u_n - y\| = 0. \tag{3.6}$$

Since the limit y of any subsequence u_n is the same, the whole sequence u_n converges to y. Thus a quasi-solution exists. Hence lemma 5 is proved. □

It remains to be proved the uniqueness and its continuous dependence on f of the quasi-solution.

THEOREM 3. *If A is linear, bounded and injective operator, K is a convex compactum and the functional $F(u)$ in minimization problem (3.1) is strictly convex, then for any f, the quasi-solution exists, is unique, and depends on f continuously.*

Proof:
The following lemmas, are needed for the proof of theorem (3).

LEMMA 6. *Let $\inf_{u\in K}\|u-f\| := dist(f,K) := m(f)$. Then there exists a unique element $u \equiv u(f) := P_K f \in K$ called the metric projection of f onto K such that $\|P_K f - f\| = dist(f,K)$.*

Proof: *Existence of $P_K f$*: Let u_n be a minimizing sequence in K, $\|u_n - f\| \longrightarrow m(f)$, Let $n \longrightarrow \infty$. Then there exists a convergent subsequence in K, which we again denote by u_n, such that $u_n \longrightarrow u \in K$. Thus, $\|u-f\| = m(f)$. So $u = P_K f$.

Uniqueness of $P_K f$: Suppose there exists u, v which are distinct metric projections. Then $m(f) = \|u-f\| = \|v-f\| \leq \|w-f\|$ for all $w \in K$. Since K is convex, $\frac{u+v}{2} \in K$. This implies that

$m(f) \leq \|\frac{u+v}{2} - f\| = \|\frac{u-f+v-f}{2}\| \leq \frac{\|u-f\|+\|v-f\|}{2} = m(f)$.

So,
$\|\frac{u+v}{2} - f\| = m(f)$.

Thus,
$\|u-f\| = \|v-f\| = \|\frac{u-f+v-f}{2}\|$.

Since X is strictly convex, it follows that $(u-f) = \lambda(v-f)$. Since $\|u-f\| = \|v-f\|$, $\lambda = +1, -1$. If $\lambda = 1$, then $u = v$ which is a contradiction. If $\lambda = -1$, then $f = \frac{u+v}{2}$. Since K is convex, this implies that $f \in K$, this gives that $P_K f = f$ which is a contradiction. Thus P_K is a bijective mapping onto K. Hence Lemma 6 is proved. □

LEMMA 7. *$dist(f,K)$ is a continuous function of f.*

Proof: Let $dist(f,K) := m(f)$. Suppose $f \to g$. Then to prove that $m(f) \to m(g)$. Let $u(f) = P_K f \in K$ and $u(g) = P_K g \in K$. (f and g are arbitrary they need not be in K).
So

$\|u(f) - f\| = inf_{u\in K}\|u-f\|$ and $\|u(g) - g\| = inf_{u\in K}\|u-g\|$.

So,
$m(f) = \|u(f)-f\| \leq \|u(g)-f\| \leq \|u(g)-g\| + \|g-f\|$.

Hence, $m(f) - m(g) \leq \|g-f\|$.
Similarly,
$m(g) - m(f) \leq \|g-f\|$.
Thus,
$|m(f) - m(g)| \leq \|f-g\|$.
Hence lemma 7 is proved. □

LEMMA 8. *$P_K f$ is a continuous function of f (in a strictly convex Banach space).*

Proof: Suppose there is a sequence $f_n \to g$. Then to prove that
$\|u(f_n) - u(g)\| \to 0 \quad as \quad n \to \infty.$ (*)

Suppose (*) is not true, so that there is a sequence u_n in K which does not satisfy (*). Since K is a compactum, there exists a subsequence $u_{nk} \in K$ of u_n, which is denoted again by u_n such that,
$\|u_n - u(g)\| \geq \epsilon > 0.$
Also, since K is closed, $u_n \to v \in K$, so that
$\|v - u(g)\| \geq \epsilon > 0.$ (**)
$\|u(g) - g\| \leq \|v - g\|.$ (***)

Now,
$\|v - g\| \leq \|v - u_n\| + \|u_n - f_n\| + \|f_n - g\|.$

By lemma 7, since $f_n \to g$, $\|u_n - f_n\| = m(f_n) \to m(g)$.

Also we have,
$\|v - u_n\| \to 0$ and $\|f_n - g\| \to 0.$

Thus,
$\|v - g\| \leq m(g) = \|u(g) - g\|.$ (****)

So, by inequalities (***) and (****),
$\|u(g) - g\| = \|v - g\|.$
This by uniqueness implies that $v = u(g)$. This contradicts inequality (**). Hence lemma 8 is proved. □

Lemma 9. *If A is a closed (possibly non-linear) injective map over a compactum $K \subset X$ onto AK, then A^{-1} is a continuous map of AK onto K.*

Proof: Let $f_n = Au_n$, where the sequences $u_n \in K$, and $f_n \in AK$. Assume that $f_n \to f$. Then to prove that $f \in AK$, that is to prove that there exists a $u \in K$ such that $u_n = A^{-1}f_n \to u = A^{-1}f$. Since K is a compactum, and since $u_n \in K$, there exists a convergent subsequence, which is again denoted by $u_n \in K$ such that $u_n \to u$. Since K is a compactum, it is closed, so $u \in K$. Because any convergent subsequence of u_n converges to a unique limit u, implies that the whole sequence converges to u. Since $u_n \to u$, $f_n = Au_n \to f$ and A is closed, therefore, $Au = f$. Since A is injective, this implies that $u = A^{-1}f$. Hence lemma 9 is proved. □

Proof of continuous dependence on f in theorem (3):

Existence of quasi-solution is proved in *lemma (5)*. Since K is convex and A is linear, so AK is convex. Since AK is convex and F is strictly convex, by lemma (6), $P_{AK}f$ exists and is unique. By lemma (8), $P_{AK}f$ depends on f continuously. Let $Au = P_{AK}f$. Since A is injective $u = A^{-1}P_{AK}f$ is uniquely defined and by lemma (9), depends continuously on f. Thus theorem (3) is proved. □

Remark:

By theorem (3), if K is a convex compactum of X which contains the solution u to equation (1.1), if A is an injective linear bounded operator, and F is strictly convex, then $u_\delta = A^{-1}P_{AK}f_\delta$ satisfies $\|u_\delta - u\| \to 0 \quad as \quad \delta \to 0$. The function u_δ can be found as the unique solution to the minimization problem (3.1) with f_δ in place of f. Further instead of assuming operator A to be bounded, A can be assumed to be closed, since a bounded operator defined everywhere is closed.

4. Iterative regularization method

Consider the operator equation (1.1) which has to be solved, where $A : H \longrightarrow H$ is assumed to be a bounded, linear injective operator on a Hilbert space H with A^{-1} unbounded. So the problem is ill-posed. The data are the elements $\{\delta, A, f_\delta\}$, where the noise level $\delta > 0$ is given such that estimate (2.1) holds, i.e., f_δ is the $\delta-$approximation of f, where $f \in R(A)$. Let $Au = f$ be solvable and let y be its minimal-norm solution. The problem is: given $\{\delta, A, f_\delta\}$, find the stable solution u_δ such that error estimate (2.3) holds. Let

$$Bu = q := A^* f, \quad where \quad B = A^* A \geq 0. \tag{4.1}$$

Let q_δ be given in place of q. Since $Au = f$ is solvable, it is equivalent to $Bu = q$. Since A is injective, B is also injective. Assume without loss of generality $\|A\| \leq 1$, which implies that $\|A^*\| \leq 1$. Since $\|f - f_\delta\| \leq \delta$ we obtain, $\|q - q_\delta\| \leq \|A^*\|\delta$, hence we obtain

$$\|q - q_\delta\| \leq \delta. \tag{4.2}$$

Consider the iterative process:

$$u_{n+1} = u_n - \mu(Bu_n - q), \quad 0 < \mu < \frac{1}{\|B\|}, \quad u(0) = u_0 \perp N(A). \tag{4.3}$$

For example one may take $u_0 = 0$. We obtain the following result:

Lemma 10. *Assume that equation (1.1) is solvable, and that y is its minimal-norm solution. Then*

$$lim_{n\to\infty} u_n = y. \tag{4.4}$$

Proof

We note that from equation (4.3),

$$y = y - \mu(By - q). \tag{4.5}$$

Denote $u_n - y := \gamma_n$. Subtracting equation (4.5) from equation (4.3) and using induction,
we obtain

$\gamma_{n+1} = \gamma_n - \mu B\gamma_n = (I - \mu B)\gamma_n = ... = (I - \mu B)^{n+1}\gamma_0$

with $\gamma_0 = u_0 - y, \quad \gamma_0 \perp N(A)$. Since, $0 < (1 - \mu\lambda) < 1$, for all $\lambda \in (0, \|B\|)$, we have,

$\|\gamma_n\|^2 = \int_0^{\|B\|} |1 - \mu\lambda|^{2n} d(E_\lambda\gamma_0, \gamma_0) = \int_0^\epsilon + \int_\epsilon^{\|B\|} := I_1 + I_2.$

If $\epsilon \leq \lambda \leq \|B\|$ then $1 - \mu\lambda \leq 1 - \mu\epsilon < 1$. Denote
$p := 1 - \mu\epsilon, 0 < p < 1$.
Then
$I_2 \leq p^{2n} \longrightarrow 0, \quad n \longrightarrow \infty$.
So
$I_2 \longrightarrow 0, \quad n \longrightarrow \infty$.
Since $1 - \mu\lambda < 1$,
$I_1 \leq \int_0^\epsilon d(E_\lambda\gamma_0, \gamma_0) \longrightarrow 0, \quad \epsilon \longrightarrow 0$.
Since $\gamma_0 \perp N(B) = N(A)$, $\|\gamma_n\| \longrightarrow 0$, as $n \longrightarrow \infty$. Hence (4.4) holds. □

Now, we shall prove the main theorem,

Theorem 4. *Suppose that A is a linear, bounded, injective operator on a Hilbert space with A^{-1} unbounded satisfying the equation $Au = f$. If q_δ is given such that*

$\|q_\delta - q\| \leq \delta$, *then one can use the iterative process (4.3) with* q_δ *in place of* q *for constructing a stable approximation of the solution* y.

Proof: By the iterative process, (4.3)

$$u_{n+1,\delta} = u_{n,\delta} - \mu(Bu_{n,\delta} - q_\delta), \quad u(0) = u_0. \tag{4.6}$$

From equation (4.5) one has:
$y = y - \mu(By - q)$.
Denote $u_{n,\delta} - y := \gamma_{n,\delta}$.
Then subtracting equation (4.5) from equation (4.6),

$$\gamma_{n+1,\delta} = \gamma_{n,\delta} - \mu B\gamma_{n,\delta} + \mu(q_\delta - q), \quad \gamma_0 = u_0 - y. \tag{4.7}$$

So that by induction,
$\gamma_{n,\delta} = (I - \mu B)^n \gamma_0 + \sum_{j=0}^{n-1} (\mu B)^j \mu(q_\delta - q), \quad \gamma_0 = u_0 - y.$
Thus,

$$\gamma_{n,\delta} = \gamma_n + \sum_{j=0}^{n-1} (\mu B)^j \mu(q_\delta - q), \quad \gamma_0 = u_0 - y. \tag{4.8}$$

Since, $\|\mu B\| \leq 1$ and by using (4.2), we obtain
$\|\gamma_{n,\delta}\| \leq \|\gamma_n\| + n\mu\delta, \quad n \geq 1$
It is already proved in lemma (10), that $\|\gamma_n\| \longrightarrow 0 \quad as \quad n \longrightarrow \infty$. Hence $\|\gamma_{n,\delta}\| \longrightarrow 0$ as $\delta \longrightarrow 0$. Thus theorem 4 is proved. □

Remark: In this method the regularization parameter is the stopping rule, $n(\delta)$, the number of iterations and can be found by solving the minimization problem
$\|\gamma_{(n)}\| + n\mu\delta = min. \longrightarrow 0$ as $\delta \longrightarrow 0, \quad n \geq 1$ and $n(\delta) \longrightarrow \infty$ as $\delta \longrightarrow 0$.

5. Dynamical systems method

In this section we study dynamical systems method for solving linear and non-linear ill-posed problems in a real Hilbert space H. The DSM for solving operator equations consists of a construction of a Cauchy problem, which has a unique global solution for an arbitrary initial data, this solution tends to a limit as time tends to infinity, and this limit is the stable solution of the given operator equation. This method can be used for solving well-posed problems also. Our discussion is based on the paper by A.G.Ramm [**2**].

Consider an operator equation

(5.1) $$F(u) := Bu - f = 0, \quad f \in H$$

where B is a linear or non-linear operator in a real Hilbert space H. We make the following assumptions.

Assumption 1 Assume that F has two Fréchet $u_{\alpha,\delta}$ derivatives: $F \in C^2_{loc}$, i.e.,

(5.2) $$\sup_{u \in B(u_0,R)} \|F^{(j)}(u)\| \le M_j(R), \quad j = 0, 1, 2$$

where $B(u_0, R) := \{u : \|u - u_0\| \le R\}$, u_0 is arbitrary fixed element in H and $R > 0$ is arbitrary and $F^{(j)}(u)$ is the j-th Fréchet derivative of $F(u)$.

Assumption 2 Assume that there exists a solution $y \in B(u_0, R)$ (not necessarily unique globally) to equation (5.1):

(5.3) $$F(y) = 0$$

Problem (5.1) is called *well-posed* if $F'(u)$ is a bounded invertible linear operator, i.e., if $[F'(u)]^{-1}$ exists and if the estimate

(5.4) $$\sup_{u \in B(u_0,R)} \|[F'(u)]^{-1}\| \le m(R),$$

Otherwise, it is called *ill-posed*.

Let $\dot{u}$ denote time-derivative. Consider the Cauchy problem (dynamical system):

(5.5) $$\dot{u} = \mathbf{\Phi}(t, u); \quad u(0) = u_0$$

where $\mathbf{\Phi}$ is a non-linear operator, which is locally Lipschitz with respect to $u \in H$ and continuous with respect to $t \ge 0$, so that the Cauchy problem (5.5) has a unique local solution. The operator $\mathbf{\Phi}$ is chosen such that the following properties hold:

(1) There exists unique global solution $u(t)$ to the Cauchy problem (5.5). (Here global solution means the solution defined for all $t > 0$.)
(2) There exists $u(\infty) := \lim_{t\to\infty} u(t)$.
(3) and finally this limit solves equation (5.1): $F(u(\infty)) = 0$.

Problem (5.1) with noisy data f_δ, $\|f_\delta - f\| \le \delta$, given in place of f, generates the problem:

(5.6) $$\dot{u}_\delta = \mathbf{\Phi}_\delta(t, u_\delta), \quad u_\delta(0) = u_0,$$

The solution u_δ to problem (5.6), calculated at $t = t_\delta$, where t_δ is suitably chosen, satisfies the error estimate

(5.7) $$\lim_{\delta\to 0} \|u_\delta(t_\delta) - y\| = 0.$$

The choice of t_δ with this property is called the *stopping rule* and is the regularization parameter in DSM method. One has usually $\lim_{\delta\to 0} t_\delta = \infty$.

Dynamical systems method can be used to solve ill-posed and also well-posed problems. In this report we are interested in discussing solving linear ill-posed problems by DSM. One can also find in A.G.Ramm's paper [**2**], a discussion of DSM for solving well-posed problems, nonlinear ill-posed problems with monotone and non-monotone operators and the recent development of the theory of DSM.

CHAPTER 3

Dynamical systems method for linear problems

In this section, for linear solvable ill-posed problem $Au = f$, with bounded linear operator $\|A\| < 1$, DSM is justified and a stable approximation of the minimal norm solution to ill-posed problem with noisy data f_δ, $\|f_\delta - f\| \le \delta$ is constructed. This section is based on paper [**2**].

Assume that (2.24) and (2.25) holds and (2.26) fails so the problem is ill-posed. Consider the equation

$$Au = f \tag{0.8}$$

where $f \in R(A)$ is arbitrary. Let us assume the following

Assumptions(*).

(1) Let A be a linear, bounded operator in a Hilbert space H, defined on all of H, the range $R(A)$ is not closed, so that A^{-1} is unbounded. So problem (0.8) is an ill-posed problem. Let f_δ be given in place of f, $\|f - f_\delta\| \le \delta$.

(2) Equation (0.8) is solvable (possibly non-uniquely). Let y be the minimal-norm solution to equation (0.8), $y \perp N(A)$, where $N(A) := \{v : Av = 0\}$ is the null-space of A.

Let $B = A^*A \ge 0$ and $q := A^*f$, A^* is the adjoint of A. Then we obtain the normal equation,

$$Bu = q. \tag{0.9}$$

We know that if equation (0.8) is solvable then it is equivalent to equation (0.9) with q_δ given in place of q. Without loss of generality assume $\|A\| \le 1$, so $\|A^*\| \le 1$ and $\|B\| \le 1$. Then $\|q - q_\delta\| = \|A^*(f - f_\delta)\| \le \|A^*\|\delta \le \delta$ and $y \perp N(B)$. Let $\epsilon(t) > 0$, be a continuous, monotonically decaying function decaying to zero function on $\mathbf{R}_+$ such that $\int_0^\infty \epsilon ds = \infty$. Let

$$F(u) := Bu - q = 0 \tag{0.10}$$

then $F^{'}(u) = B$. Consider the Cauchy problem

$$\dot{u} = \mathbf{\Phi}(t, u), \quad u(0) = u_0. \tag{0.11}$$

$\mathbf{\Phi(t, u)} := -[F^{'}(u) + \epsilon(t)]^{-1}[F(u) + \epsilon(t)u] = -[B + \epsilon(t)]^{-1}[Bu - q + \epsilon(t)u]$.

Thus from the Cauchy problem (0.11), the DSM for solving equation (0.10) is solving the Cauchy problem

$$\dot{u} = \mathbf{\Phi}(t, u) = -u + [B + \epsilon(t)]^{-1}q, \quad u(0) = u_0. \tag{0.12}$$

with

$$\mathbf{\Phi}_\delta = -u_\delta + [B + \epsilon(t)]^{-1}q_\delta. \tag{0.13}$$

We now prove the main theorem of this section: *Given noisy data f_δ, every linear ill-posed problem (0.8) under the assumptions (*) can be stably solved by the DSM.*

THEOREM 5. *Assume (*), and let $B := A^*A$, $q := A^*f$. Assume $\epsilon(t) > 0$ to be a continuous, monotonically decaying to zero function on $[0, \infty)$ such that $\int_0^\infty \epsilon ds = \infty$. Then we have the following results.*

(1) *For any $u_0 \in H$, the Cauchy problem (0.12) has a unique global solution $u(t)$, (the initial approximation u_0 need not be close to the solution $u(t)$ in any sense).*

(2) *There exists $\lim_{t\to\infty} u(t) = u(\infty) = y$, and y is the unique minimal-norm solution to equation (0.8). $Ay = f, y \perp N$, and $\|y\| \le \|z\|$, for all $z \in N := \{z : F(z) = 0\}$.*

(3) *If f_δ is given in place of f, $\|f - f_\delta\| \le \delta$, then there exists a unique global solution $u_\delta(t)$ to the Cauchy problem*

$$\dot{u}_\delta = \mathbf{\Phi}_\delta(t, u_\delta) = -u_\delta + [B + \epsilon(t)]^{-1} q_\delta, \quad u_\delta(0) = u_0 \tag{0.14}$$

with $q_\delta := A^* f_\delta$.

(4) *There exists t_δ, such that it satisfies the error estimate*

$$\lim_{\delta\to 0} \|u_\delta(t_\delta) - y\| = 0, \quad \lim_{\delta\to 0} t_\delta = \infty. \tag{0.15}$$

The choice of t_δ with this property is the stopping rule. This t_δ can be for example chosen by a discrepancy principle or as a root of the equation

$$2\sqrt{\epsilon(t)} = \delta^b, \quad b \in (0, 1). \tag{0.16}$$

Proof: Since the Cauchy problem (0.11) is linear, its solution can be written by an explicit analytic formula

$$u(t) = u_o e^{-t} + \int_0^t e^{-(t-s)} [B + \epsilon(s)]^{-1} By ds. \tag{0.17}$$

Taking limit as $t \to \infty$ in (0.17) and applying L'Hospital's rule, to the second term in the right hand side of equation (0.17), we obtain,

$\lim_{t\to\infty} \frac{\int_0^t e^s [B+\epsilon(s)]^{-1} By ds}{e^t} = \lim_{t\to\infty} [B + \epsilon(t)]^{-1} By,$

provided only that $\epsilon(t) > 0$ and $\lim_{t\to\infty} \epsilon(t) = 0$.

Since $y \perp N = N(B) = N(A)$,

$$\lim_{\epsilon\to 0} [B + \epsilon]^{-1} By = \lim_{\epsilon\to 0} \int_0^{\|B\|} \frac{\lambda}{\lambda + \epsilon} dE_\lambda y = \int_0^{\|B\|} dE_\lambda y = y, \tag{0.18}$$

by the spectral theorem and by the Lebesgue dominated convergence theorem, where E_λ is the resolution of the identity corresponding to the self-adjoint operator B, λ is taken over the spectrum of B, and $\lim_{\epsilon\to 0} \frac{\lambda}{\lambda+\epsilon} = 1$, for $\lambda > 0$ and $= 0$, for $\lambda = 0$. Thus from equations (0.17) and (0.18) there exists $u(\infty) = \lim_{t\to\infty} u(t) = y$ with $Ay = f$.

Denote $\eta(t) := \|u(t) - y\|$, then $\lim_{t\to\infty} \eta(t) = 0$. In general, the rate of convergence of η to zero can be arbitrarily slow for a suitably chosen f. Under an additional a priori assumption of f (for example, the source type assumptions), this rate can be estimated.

Proof of results 3 and 4: Derivation of the stopping rule.

Consider the Cauchy problem with noisy data. Suppose f_δ is given, with $\|f_\delta - f\| \le \delta$, then $\|q_\delta - q\| \le \delta$. We require the following lemma for the proof:

Lemma 11.

$$\|[B+\epsilon]^{-1}A^*\| \le \frac{1}{2\sqrt{\epsilon}} \tag{0.19}$$

Proof of lemma: We have $B := A^*A \ge 0$. Denote $Q := AA^* \ge 0$, then $\|Q\| \le 1$. We have

$[B+\epsilon I]^{-1}A^* = A^*[Q+\epsilon I]^{-1}$.

So,

$\|[B+\epsilon I]^{-1}A^*\| = \|A^*[Q+\epsilon I]^{-1}\| = \|UQ^{1/2}[Q+\epsilon I]^{-1}\|$,

(U being the isometry operator ($\|Uf\| = \|f\|$), and $A = U|A| = U(A^*A)^{1/2}$,polar representation of the linear operator).

$= \|Q^{1/2}[Q+\epsilon I]^{-1}\|$, (since $\|U\| = 1$.) $= \| \int_0^1 \frac{\lambda^{1/2}}{\lambda+\epsilon} dE_\lambda\| = \sup_{0<\lambda\le 1} \frac{\lambda^{1/2}}{\lambda+\epsilon} = \frac{1}{2\sqrt{\epsilon}}$.

(by the spectral theorem: $\phi(A) = \int_{-\infty}^{\infty} \phi(\lambda) dE_\lambda$, $\|\phi(A)\| = \sup_\lambda |\phi(\lambda)|$, λ is taken over the spectrum of A). $\square$

By triangle inequality,

$\|u_\delta(t) - y\| = \|u_\delta(t) - u(t) + u(t) - y\| \le \|u_\delta(t) - u(t)\| + \|u(t) - y\| = \|u_\delta(t) - u(t)\| + \eta(t)$.

$\|u_\delta(t) - u(t)\| = \| \int_0^t e^{-(t-s)}[B+\epsilon(s)]^{-1}(q_\delta - q)ds\| \le \int_0^t e^{-(t-s)} \frac{\delta}{2\sqrt{\epsilon}}$, by (0.19) $\le \frac{\delta}{2\sqrt{\epsilon}}$, since $\int_0^t e^{-(t-s)}ds = 1 - e^{-t} \le 1$.

So,

$$\|u_\delta(t) - u(t)\| \le \frac{\delta}{2\sqrt{\epsilon}}. \tag{0.20}$$

Thus, $\|u_\delta(t) - y\| \le \frac{\delta}{2\sqrt{\epsilon}} + \eta(t)$.

We have already proved that $\lim_{t\to\infty} \eta(t) = 0$. Choose, $t = t_\delta$, satisfying equation (0.16) then this particular choice of t_δ satisfies the error estimate (0.15) and $\|u_\delta(t) - u(t)\| \longrightarrow 0 \quad as \quad \delta \longrightarrow 0$. If the decay rate of $\eta(t)$ is known, a more efficient stopping rule can be obtained by choosing, $t = t_\delta$ such that it satisfies the minimization problem

$\frac{\delta}{2\sqrt{\epsilon}} + \eta(t_\delta) = \min. \longrightarrow 0$ as $\delta \longrightarrow 0$. $\square$

Remarks

Remark 1: *Discrepancy principle for the DSM.*

Choose the stopping time t_δ as the unique solution to the equation: $\|Au_\delta(t) - f_\delta\| = C\delta$ where C=constant> 1, where it is assumed that $\|f_\delta\| > \delta$. In-addition, we assume that $f_\delta \perp N(A^*)$, so that C=1, then the equation is:

$$\|A[B+\epsilon(t)]^{-1}A^*f_\delta - f_\delta\| = \delta. \tag{0.21}$$

Then this t_δ satisfies the error estimate (0.15). One can find detailed discussion of this in A.G.Ramm's paper [**5**].

Remark 2: *Choosing scaling parameter $\epsilon(t)$.*

We can choose the scaling parameter as large as we wish. In particular we can choose

$$\epsilon(t) = \frac{c_1}{(c_0+t)^b} \tag{0.22}$$

where, $0 < b < 1, c_1, c_2$ are positive constants.

Bibliography

[1] Ramm,A.G., *Regularization of ill-posed problems with unbounded operators,* J.Math.Anal.Appl. 271, (2002), 547-550.

[2] Ramm,A.G., *Dynamical systems method for solving operator equations,* Communic. in Nonlinear Sci. and Numer. Simulation, 9, N2, (2004), 547-550.

[3] Ramm,A.G., *On a new notion of regularizer,* J.Phys.A. 36, (2003), 2191-2195.

[4] Ramm,A.G., *Global convergence for ill-posed equations with monotone operators: the dynamical systems method,* J.Phys A, 36, (2003), L249-L254.

[5] Ramm,A.G., *Lecture Notes,Math 994, Ill-Posed And Inverse Problems,* Department of Mathematics, Kansas State University, (Fall 2003, Spring 2004).

[6] Ramm,A.G., *Random Fields Estimation Theory,* Longman/Wiley, NY,(1990), 106-134.

[7] Ramm,A.G., *On the discrepancy principle,* Nonlinear Funct. Anal. Appl., 8, N2, (2003), 307-312.

[8] Morozov,V.A., *Methods For Solving Incorrectly Posed Problems,* Springer Verlag, NY, (1984).

[9] Kaplan.A. and Tichatschke.R., *Stable Methods For Ill-Posed Variational Problems,* Akademie Verlag, Berlin, (1994).

[10] Lavrent'ev.M.M., Romanov.V.G., and Shishatskii.S.P., *Ill-Posed problems Of Mathematical Physics and Analysis,* Translations Of Mathematical Monographs, 64, AMS., Providence, Rhode Island (1986).

[11] Gorenflo.R. and Hoffmann.K.H., *Applied Nonlinear Functional Analysis, Variational Methods And Ill-Posed Problems,* VerlagPeter lang, Frankfurt, (1983).

[12] Hammerlin.G. and Hoffmann.K.H., *Improperly Posed Problems And Their Numerical Treatment,* International Series Of Numerical Mathematics, 63, Birkhauser,(1983).

[13] Ramm,A.G., *Iterative Methods For Calculating Static Fields And Wave Scattering By Small Bodies,* Springer Verlag, NY., (1982).

[14] Ramm.A.G. and Katsevich.A.I., *The Radon Transform And Local Tomography,* CRC., NY., (1996).

[15] Engl.H.W. and Groetsch.C.W., *Inverse And Ill-Posed Problems,* Notes And Reports In Mathematics In Science And Engineering, 4, Academic Press, Boston (1987).

[16] Ramm.A.G., *Discrepancy principle for the dynamical systems method*, Communic. in Nonlinear Sci. and Numer.Simulation, (to appear).

[17] Ramm.A.G, *On numerical differentiation*, Mathematics,Izvestija VUZov, 11, (1968), 131-135.